Time scale of thermal conduction

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The importance of thermal conduction

• Energy consumption:
  • transfer through conduction

• Energy generation:
  • Thermoelectric effect (Seebeck effect)
  • Spin Seebeck effect
Basics of thermal conduction
Heat Conductivity in Solids (an example for irreversibility)

Remember: Heat is an energy transferred from one system to another because of temperature difference

\[ T_1 > T_2 \]

Heat flows from \( \Sigma_1 \) to \( \Sigma_2 \)
Heat reservoir $\Sigma_1$ > Heat reservoir $\Sigma_2$ *(in the textbook $T_2 > T_1$)*

Heat transfer per time interval through homogeneous solid object:

$$\frac{Q}{\Delta t} = \frac{K}{L} (T_1 - T_2) A$$

where

- $K$: thermal conductivity of the rod
- $A$: cross-section of the rod

Example of irreversible process: heat conduction as a non-equilibrium process:

$T_1$ > $T_2$
Diffusion (time and space dependence of matter/energy)

\[ \frac{Q}{\Delta t} = \frac{K}{L} (T_1 - T_2)A \]

\[ L \rightarrow \Delta x \]
\[ T_2 - T_1 \rightarrow \Delta T \]

\[ \frac{Q}{A\Delta t} = -K \frac{\Delta T}{\Delta x} \]

Define flux:

\[ S = \lim_{\Delta t \to 0} \frac{Q}{A\Delta t} \]

Fourier’s law

\[ S(x, t) = -K \frac{\partial T(x, t)}{\partial x} \]
Diffusion (time and space dependence of matter/energy)

\[ S(x, t) \rightarrow u(x, t), U \rightarrow S(x + \Delta x, t) \]

\( u \): energy density (energy per unit volume)

\( S \): flux (energy flow per unit area per unit time, similar to electric current)

How would energy change in the system?

\[ A[S(x + \Delta x, t) - S(x, t)]\Delta t = -[u(x, t + \Delta t) - u(x, t)]A\Delta x \]

\[ \frac{[S(x + \Delta x, t) - S(x, t)]}{\Delta x} = - \frac{[u(x, t + \Delta t) - u(x, t)]}{\Delta t} \]

Continuity equation:

\[ \frac{\partial u(x, t)}{\partial t} = - \frac{\partial S(x, t)}{\partial x} \]
**Diffusion** (time and space dependence of matter/energy)

Flux across a **boundary**

\[ S(x, t) = -K \frac{\partial T(x, t)}{\partial x} \]

Energy change in a **system**

\[ \frac{\partial u(x, t)}{\partial t} = -\frac{\partial S(x, t)}{\partial x} \]
Diffusion (time and space dependence of matter/energy)

\[ S(x, t) \rightarrow u(x, t), U \rightarrow S(x + \Delta x, t) \]

\( u : \) energy density (energy per unit volume)

\( S : \) flux (energy flow per unit area per unit time, similar to electric current)

To study temperature distribution.

\[ U = U_0 + \left( \frac{\partial U}{\partial T} \right)_V \Delta T = U_0 + Mc_V^M \Delta T \]

\[ u = \frac{U(t)}{V} = \frac{U_0}{V} + \frac{Mc_V^M \Delta T}{V} = u_0 + \rho c_V^M \Delta T \]

\[ u - u_0 = \Delta u = \rho c_V^M \Delta T \]

\[ \frac{\partial u(x, t)}{\partial t} = \rho c_V^M \frac{\partial T(x, t)}{\partial t} \]

Continuity equation:

\[ \frac{\partial S(x, t)}{\partial x} = -\rho c_V^M \frac{\partial T(x, t)}{\partial t} \]
Diffusion (time and space dependence of matter/energy)

\[ S(x, t) \rightarrow u(x, t), U \rightarrow S(x + \Delta x, t) \]

\[ x \rightarrow x + \Delta x \]

**\( S \): flux (energy flow per unit area per unit time, similar to electric current)**

**\( u \): energy density (energy per unit volume)**

\[
\frac{\partial S(x,t)}{\partial x} = -\rho c_v^M \frac{\partial T(x,t)}{\partial t} \quad (1)
\]

\[
\frac{\partial T(x,t)}{\partial t} = D \frac{\partial^2 T(x,t)}{\partial x^2} \quad (2)
\]

\[ \text{Diffusion equation} \]

\[ D = \frac{K}{\rho c_v^M} \quad \text{diffusivity} \]
Application of Diffusion Equation (Decay of a hot spot)

Diffusion equation
\[
\frac{\partial T(x, t)}{\partial t} = D \frac{\partial^2 T(x, t)}{\partial x^2}
\]

Describes how the spatial distribution evolves with time.

One of the solution (example):
\[
T(x, t) = T_0 + \frac{A}{\sqrt{t + t_0}} \exp \left( -\frac{x^2}{4D(t + t_0)} \right)
\]

Gaussian peak.
\[
T(x, 0) = T_0 + \frac{A}{\sqrt{t_0}} \exp (-\frac{x^2}{4Dt_0})
\]
Application of Diffusion Equation (Decay of a hot spot)

Spatial dependence at $t$:

$$T = T_0 + \frac{A}{w} \exp\left(-\frac{x^2}{w^2(t)}\right), \quad w(t) = \sqrt{4D(t + t_0)}$$

Spatial evolution with time:

1) Peak becomes broader

2) Total area is conserved (energy conservation $dU = dT$).

$$\frac{\partial T(x, t)}{\partial t} = D \frac{\partial^2 T(x, t)}{\partial x^2}$$
Scaling rule of the decay

\[ T(x, t) = T_0 + \frac{A}{\sqrt{t + t_0}} \exp \left( -\frac{x^2}{4D(t + t_0)} \right) \]

\[ = T_0 + \frac{A}{\sqrt{t_0}} \frac{1}{\sqrt{1 + \frac{t}{t_0}}} \exp \left( -\frac{x^2}{4Dt_0 \left( 1 + \frac{t}{t_0} \right)} \right) \]

If we use \( t_0 \) as the unit, and let \( \tau = \frac{t}{t_0} \), we get a universal relation

\[ T(x, t) = T_0 + \frac{A}{\sqrt{1 + \tau}} \exp \left( -\frac{x^2}{4D(1 + \tau)} \right) \]

So the decay scales with \( t_0 \)
Scaling rule of the decay

\[ T(x, 0) = T_0 + \frac{A}{\sqrt{t_0}} \exp\left( -\frac{x^2}{4Dt_0} \right) \]

\[ \sigma^2 = 2Dt_0 \]

Since the decay scales with \( t_0 \), it scales with the \( \sigma^2 \).
Another view using analogy to circuits

Thermal conduction is like discharging a capacitor, the time constant is

$$\tau = RC$$

For thermal conduction, we use the same formula

$$\tau = RC$$

$$R = \frac{d}{\sigma A}$$

$$C = c_M \rho V = c_M \rho A d$$

$\sigma$: thermal conductivity
$d$: thickness
$A$: area
$\rho$: density
$c_M$: specific heat

So

$$\tau = RC = \frac{d}{\sigma A} c_M \rho A d = \frac{c_M \rho}{\sigma} d^2$$

Again, this scales with width$^2$. 
Some hands-on data
A simple experiment

![Graph showing the relationship between normalized temperature change (dT) and time (dt) with two lines indicating 'dT more' and 'dT less'.]

Less

More
For the previous water cooling experiment:

\[ \frac{\tau}{d^2} = 0.003 \text{ (ns/nm}^2) \]

\[ \tau = 1080 \text{ s} \]

\[ d = 2 \text{ cm} \]
Conclusion

• We discussed the scaling rule of the thermal conduction, which is proportional to the thickness^2.

• A few examples (microscopic and macroscopic) are given, which can be unified using the scaling rule.