Transmission line calculations

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Review:

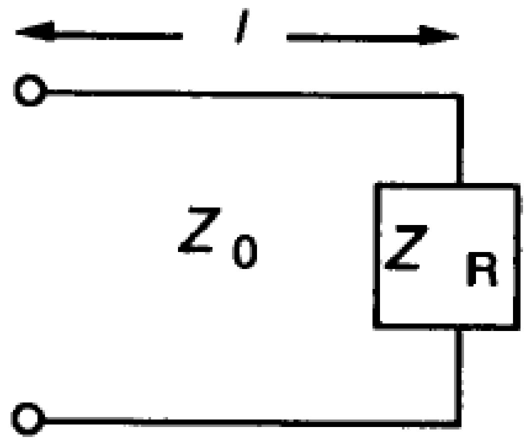
Telegrapher's equation

$$\frac{\partial i}{\partial y} = -vG - \frac{\partial v_{out}}{\partial t}C$$

$$\frac{\partial v}{\partial y} = -iR - \frac{\partial i_{in}}{\partial t}L$$

$$\longrightarrow \qquad \frac{v}{i} = Z_0 = \sqrt{\frac{R+jwL}{G+jwC}}$$

1.Line terminated with an arbitrary load



The impedance seen looking into a generalized transmission line terminated in ZR is

$$Z_{in} = Z_0 \left(\frac{Z_R \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_R \sinh \gamma l} \right) \ (\Omega)$$

Or

$$Z_{in} = Z_0 \left(\frac{Z_R \cos \beta l + j Z_0 \sin \beta l}{Z_0 \cos \beta l + j Z_R \sin \beta l} \right)$$

where

$$\alpha = 0$$

$$\beta = \frac{2\pi}{\lambda}$$

$$\gamma = j\beta = \frac{2\pi j}{\lambda}$$

Example: Quarter-wave line with open circuit load

What is the impendence looking into a lossless line one quarter of a wavelength long terminated in an open circuit?

$$l = \lambda / 4$$

$$Z_R = \infty \ \Omega$$

So from Equation (2.8.1.1):

$$Z_{in} = Z_0 \left(\frac{Z_R \cos \beta l + j Z_0 \sin \beta l}{Z_0 \cos \beta l + j Z_R \sin \beta l} \right) = Z_0 \left(\frac{\infty \cos \frac{\pi}{2} + j Z_0 \sin \frac{\pi}{2}}{Z_0 \cos \frac{\pi}{2} + j \infty \sin \frac{\pi}{2}} \right)$$

$$\boxed{Z_{in} = \frac{Z_0}{\infty} = 0 \ \Omega}$$

Example: Quarter-wave line with a generalized load

What happens when we put a general impedance ZL at the end of a quarter wavelength line?

$$l = \lambda / 4$$

$$Z_R = Z_L$$

So again using (2.8.1.1):

$$Z_{in} = Z_0 \left(\frac{Z_R \cos \beta l + j Z_0 \sin \beta l}{Z_0 \cos \beta l + j Z_R \sin \beta l} \right) = Z_0 \left(\frac{j Z_0}{j Z_L} \right) = \frac{Z_0^2}{Z_L}$$

Example: Line terminated with short circuit

If we terminate the line in a short circuit, then

$$Z_{in} = Z_{sc} = Z_0 \left(\frac{Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l} \right) = Z_0 \tanh \gamma l$$

For lossless lines, Equation (2.8.1.1) reduces to:

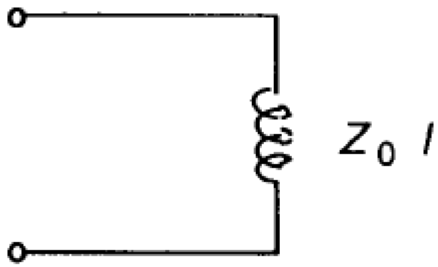
$$Z_{sc} = j\,Z_0 \tan \beta l = j\,Z_0 \tan \left(\frac{2\pi l}{\lambda} \right)$$

If $\tan x$ is now approximated with x (true for small x):

$$Z_{sc} \cong j\,Z_0\,\beta l = j\,Z_0 \left(\frac{2\pi l}{\lambda} \right) = j\,\omega\,\frac{(Z_0\,l)}{c}$$

$$\pi / 2 < \left(\frac{2\pi l}{\lambda}\right) < \pi$$

the tangent function is negative and Z_{sc} is approximately:



$$Z_{sc} \equiv -j\, Z_0\, \beta l = -j\, Z_0 \left(\frac{2\pi l}{\lambda}\right) = -j\, \omega\, \frac{Z_0\, l}{c} = \frac{1.0}{j\left(\frac{c}{\omega\, Z_0\, l}\right)}$$

which is the impedance of a capacitor of value

$$C = \frac{1.0}{\omega^2 Z_0\, l}$$

Finally, for

$$2\, \pi\, l \,/\, \lambda = \pi / 2$$

or equivalently,

$$l = \lambda / 4$$

Summarizing our results for the short-circuited line:

$$\text{“inductive”}\begin{cases} l < \dfrac{\lambda}{4} \\ \dfrac{\lambda}{2} < l < \dfrac{3\lambda}{4} \end{cases}$$

$$\text{“capacitive”}\begin{cases} \dfrac{\lambda}{4} < l < \dfrac{\lambda}{2} \\ \dfrac{3\lambda}{4} < l < \lambda \end{cases}$$

$$\text{“series LC resonator”}\left\{ l = \dfrac{n\lambda}{4} \text{ where } n \in \mathrm{I} \text{ (the integers)} \right.$$