Time scale of thermal conduction

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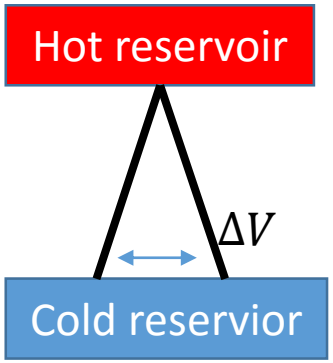
The importance of thermal conduction

- Energy consumption:
  - transfer through conduction
- Energy generation:
  - Thermoelectric effect (Seebeck effect)
  - Spin Seebeck effect

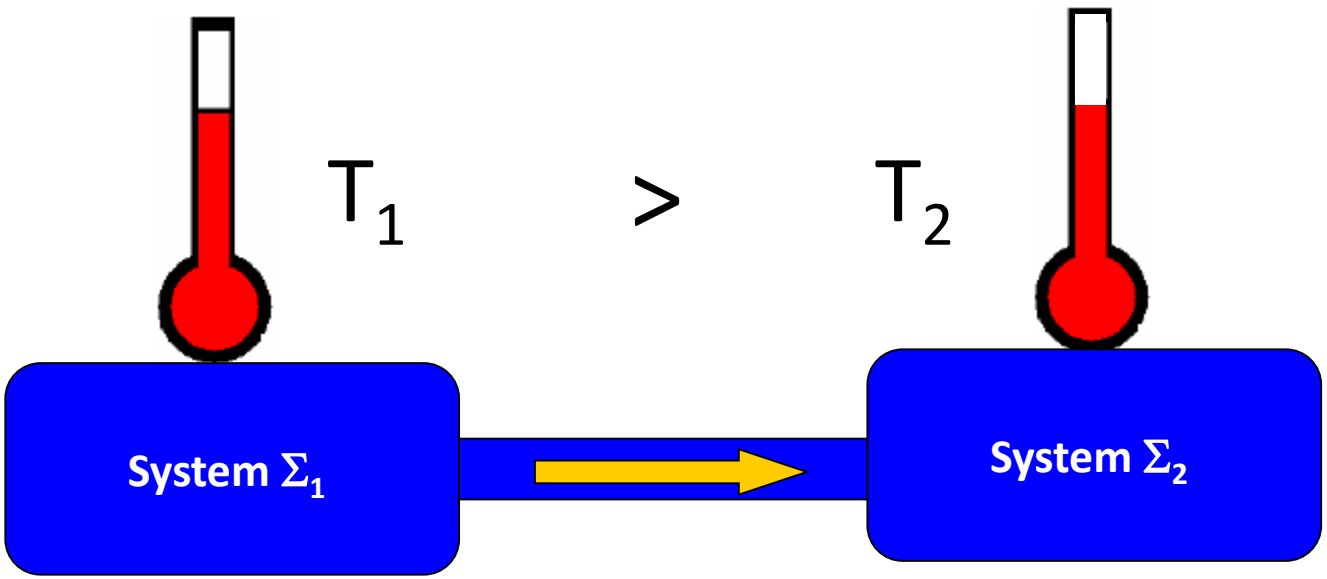


Basics of thermal conduction

Example of irreversible process: Heat conduction

Heat Conductivity in Solids (an example for irreversibility)

Remember: Heat is an energy transferred from one system to another because of temperature difference

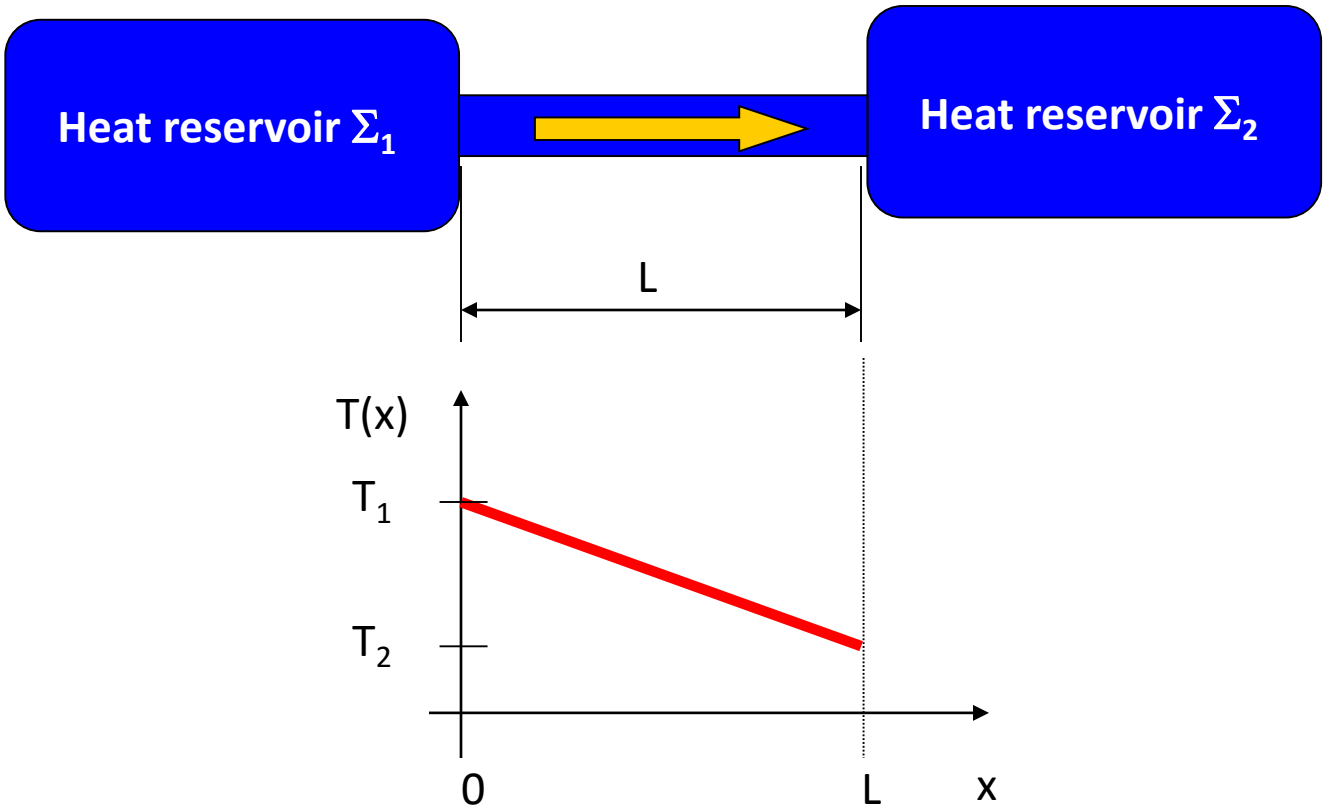


Heat Q flows from

Σ_1 to Σ_2

Example of irreversible process: heat conductions as a non-equilibrium process:

$T_1 > T_2$ *(in the textbook $T_2 > T_1$)

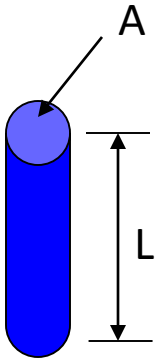


Heat transfer per time interval through homogeneous solid object:

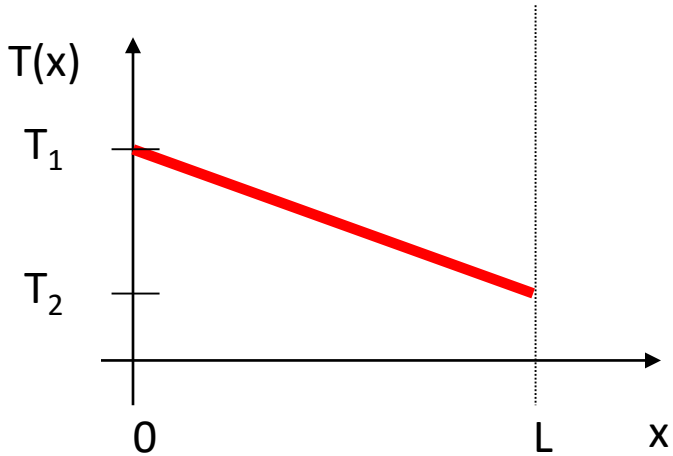
$$\frac{Q}{\Delta t} = \frac{K}{L}(T_1 - T_2)A$$

where

- K: thermal conductivity of the rod
- A: cross-section of the rod



Diffusion (time and space dependence of matter/energy)



$$\frac{Q}{\Delta t} = \frac{K}{L}(T_1 - T_2)A$$

$$L \to \Delta x$$
$$T_2 - T_1 \to \Delta T$$

$$\frac{Q}{A\Delta t} = -K\frac{\Delta T}{\Delta x}$$

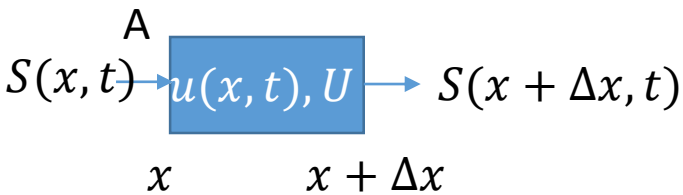
Define flux:

$$S = \lim_{\Delta t \to 0} \frac{Q}{A\Delta t}$$

Fourier's law

$$S(x,t) = -K\frac{\partial T(x,t)}{\partial x}$$

Diffusion (time and space dependence of matter/energy)



$\boldsymbol{S}$: flux (energy flow per unit area per unit time, similar to electric current)
$\boldsymbol{u}$: energy density (energy per unit volume)

How would energy change in the system?

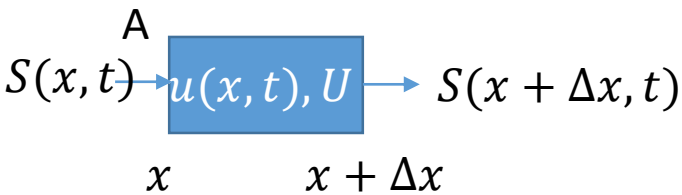
$$A[S(x+\Delta x,t)-S(x,t)]\Delta t = -[u(x,t+\Delta t)-u(x,t)]A\Delta x$$

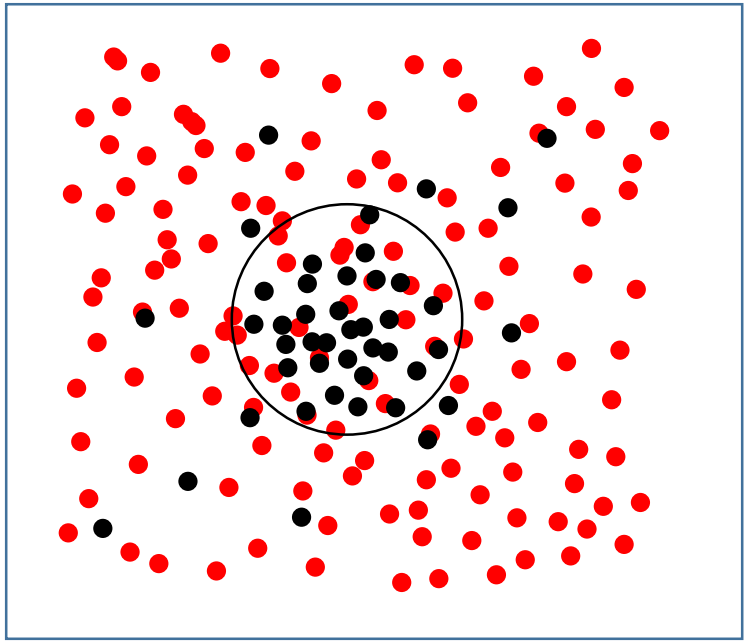
$$\frac{[S(x+\Delta x,t)-S(x,t)]}{\Delta x} = -\frac{[u(x,t+\Delta t)-u(x,t)]}{\Delta t}$$

Continuity equation:

$$\frac{\partial u(x,t)}{\partial t} = -\frac{\partial S(x,t)}{\partial x}$$

Diffusion (time and space dependence of matter/energy)





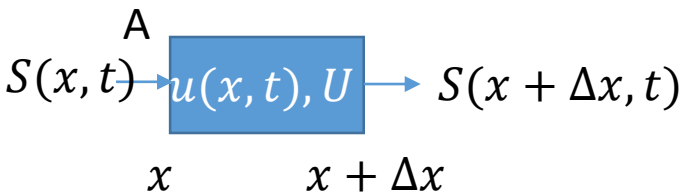
Flux across a **boundary**

$$S(x,t) = -K\frac{\partial T(x,t)}{\partial x}$$

Energy change in a **system**

$$\frac{\partial u(x,t)}{\partial t} = -\frac{\partial S(x,t)}{\partial x}$$

Diffusion (time and space dependence of matter/energy)



$\boldsymbol{S}$: flux (energy flow per unit area per unit time, similar to electric current)

$\boldsymbol{u}$: energy density (energy per unit volume)

To study temperature distribution.

$$\frac{\partial u(x,t)}{\partial t} = -\frac{\partial S(x,t)}{\partial x}$$

Continuity equation:

$$U = U_0 + \left(\frac{\partial U}{\partial T}\right)_V \Delta T = U_0 + M c_V^M \Delta T$$

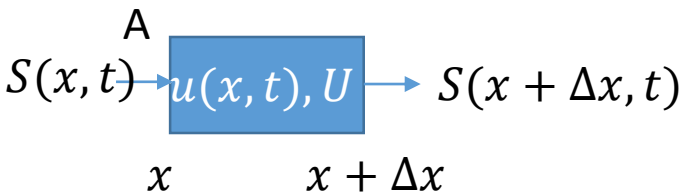
$$u = \frac{U(t)}{V} = \frac{U_0}{V} + \frac{M c_V^M \Delta T}{V} = u_0 + \rho c_V^M \Delta T$$

$$u - u_0 = \Delta u = \rho c_V^M \Delta T$$

$$\frac{\partial u(x,t)}{\partial t} = \rho c_V^M \frac{\partial T(x,t)}{\partial t}$$

$$\frac{\partial S(x,t)}{\partial x} = -\rho c_V^M \frac{\partial T(x,t)}{\partial t}$$

Diffusion (time and space dependence of matter/energy)



$\boldsymbol{S}$: flux (energy flow per unit area per unit time, similar to electric current)

$\boldsymbol{u}$: energy density (energy per unit volume)

From continuity equation

$$\frac{\partial S(x,t)}{\partial x} = -\rho c_V^M \frac{\partial T(x,t)}{\partial t} \quad (1)$$

From Fourier's law

$$S(x,t) = -K\frac{\partial T(x,t)}{\partial x} \xrightarrow{\text{Take } \frac{\partial}{\partial x}} \frac{\partial S(x,t)}{\partial x} = -K\frac{\partial^2 T(x,t)}{\partial x^2} \quad (2)$$

Diffusion equation

$$\frac{\partial T(x,t)}{\partial t} = D\frac{\partial^2 T(x,t)}{\partial x^2}$$

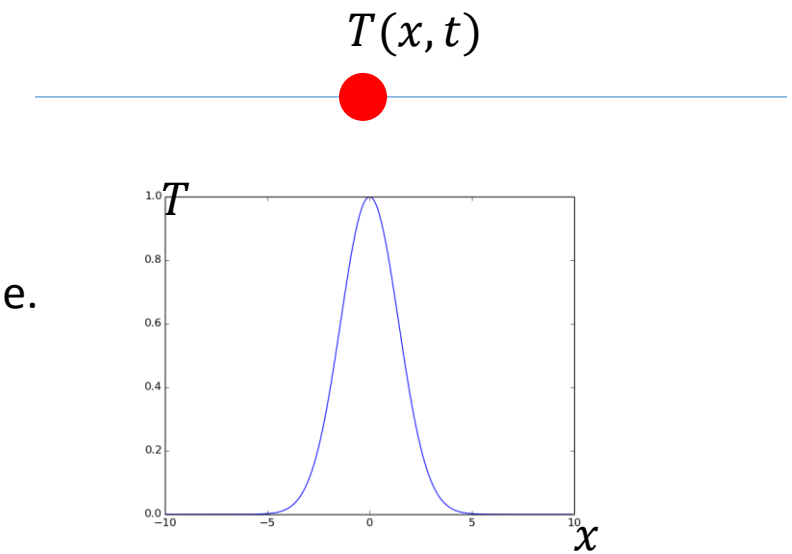
$$D = \frac{K}{\rho c_V^M}$$ diffusivity

Application of Diffusion Equation (Decay of a hot spot)

Diffusion equation

$$\frac{\partial T(x,t)}{\partial t} = D\frac{\partial^2 T(x,t)}{\partial x^2}$$

Describes how the spatial distribution evolves with time.



One of the solution (example):

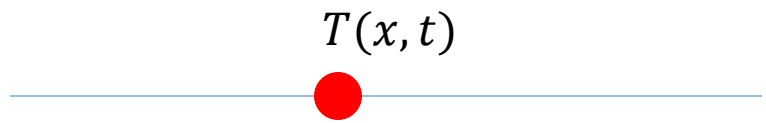
$$T(x,t) = T_0 + \frac{A}{\sqrt{t+t_0}}\exp\left(-\frac{x^2}{4D(t+t_0)}\right)$$

Gaussian peak.

$$T(x,0) = T_0 + \frac{A}{\sqrt{t_0}}\exp\left(-\frac{x^2}{4Dt_0}\right)$$

Application of Diffusion Equation (Decay of a hot spot)

Spatial dependence at t:

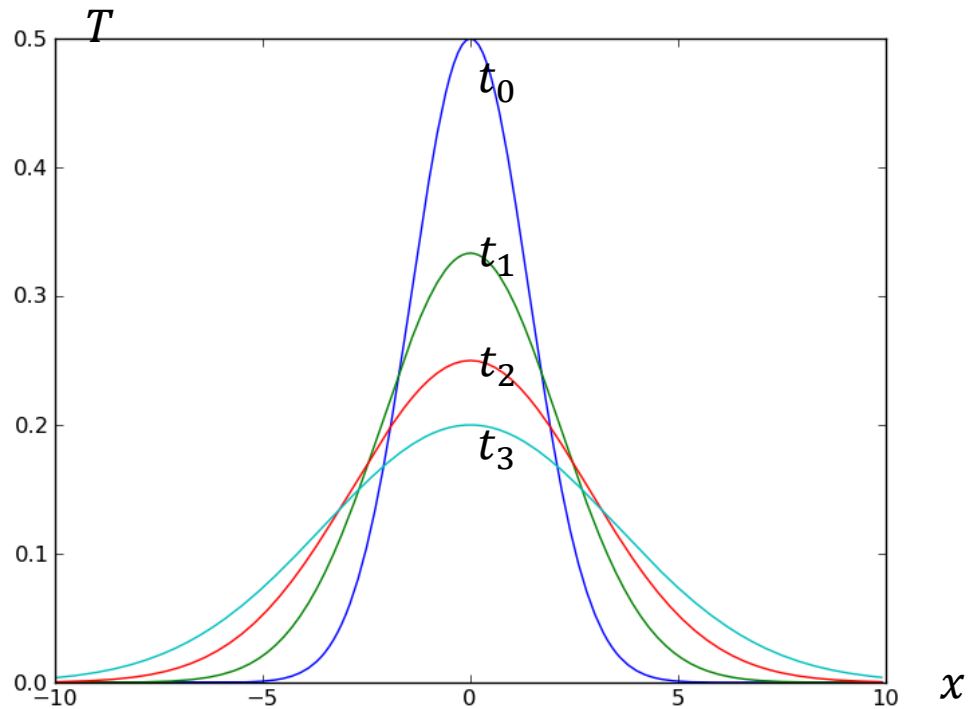


$$T = T_0 + \frac{A}{w}\exp\left(-\frac{x^2}{w^2(t)}\right),\ w(t) = \sqrt{4D(t+t_0)}$$

Spatial evolution with time:

1) Peak becomes broader

2) Total area is conserved (energy conservation $dU = dT$).



$$\frac{\partial T(x,t)}{\partial t} = D\frac{\partial^2 T(x,t)}{\partial x^2}$$

Scaling rule of the decay

$$T(x,t) = T_0 + \frac{A}{\sqrt{t+t_0}} \exp\left(-\frac{x^2}{4D(t+t_0)}\right)$$
$$= T_0 + \frac{A}{\sqrt{t_0}\sqrt{1+\frac{t}{t_0}}} \exp\left(-\frac{x^2}{4Dt_0\left(1+\frac{t}{t_0}\right)}\right)$$

If we use t_0 as the unit, and let $\tau = \frac{t}{t_0}$, we get a universal relation

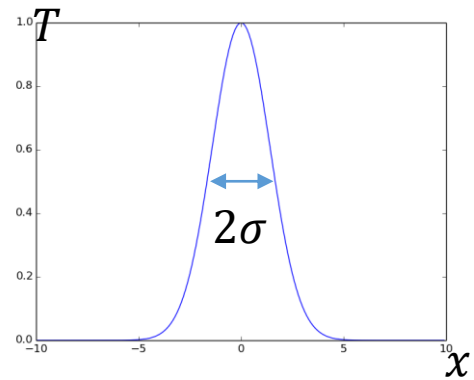
$$T(x,t) = T_0 + \frac{A}{\sqrt{1+\tau}} \exp\left(-\frac{x^2}{4D(1+\tau)}\right)$$

So the decay scales with t_0

Scaling rule of the decay

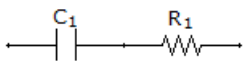
$$T(x,0) = T_0 + \frac{A}{\sqrt{t_0}}\exp(-\frac{x^2}{4Dt_0})$$

$$\sigma^2 = 2Dt_0$$



Since the decay scales with t_0, it scales with the σ^2.

Another view using analogy to circuits



Thermal conduction is like discharging a capacitor, the time constant is

$$\tau = RC$$

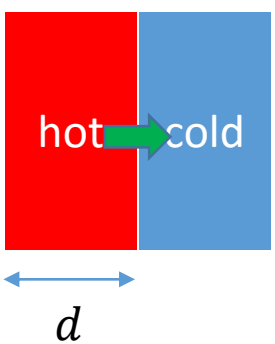
For thermal conduction, we use the same formula

$$\tau = RC$$

$$R = \frac{d}{\sigma A}$$

$$C = c_M \rho V = c_M \rho A d$$

σ: thermal conducitivity
d: thickness
A: area
ρ: density
c_M: specific heat



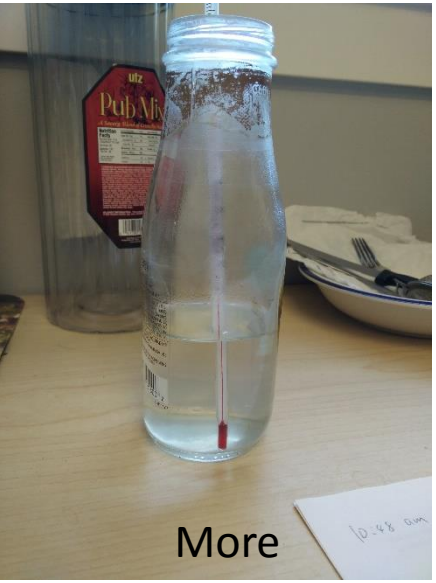
So

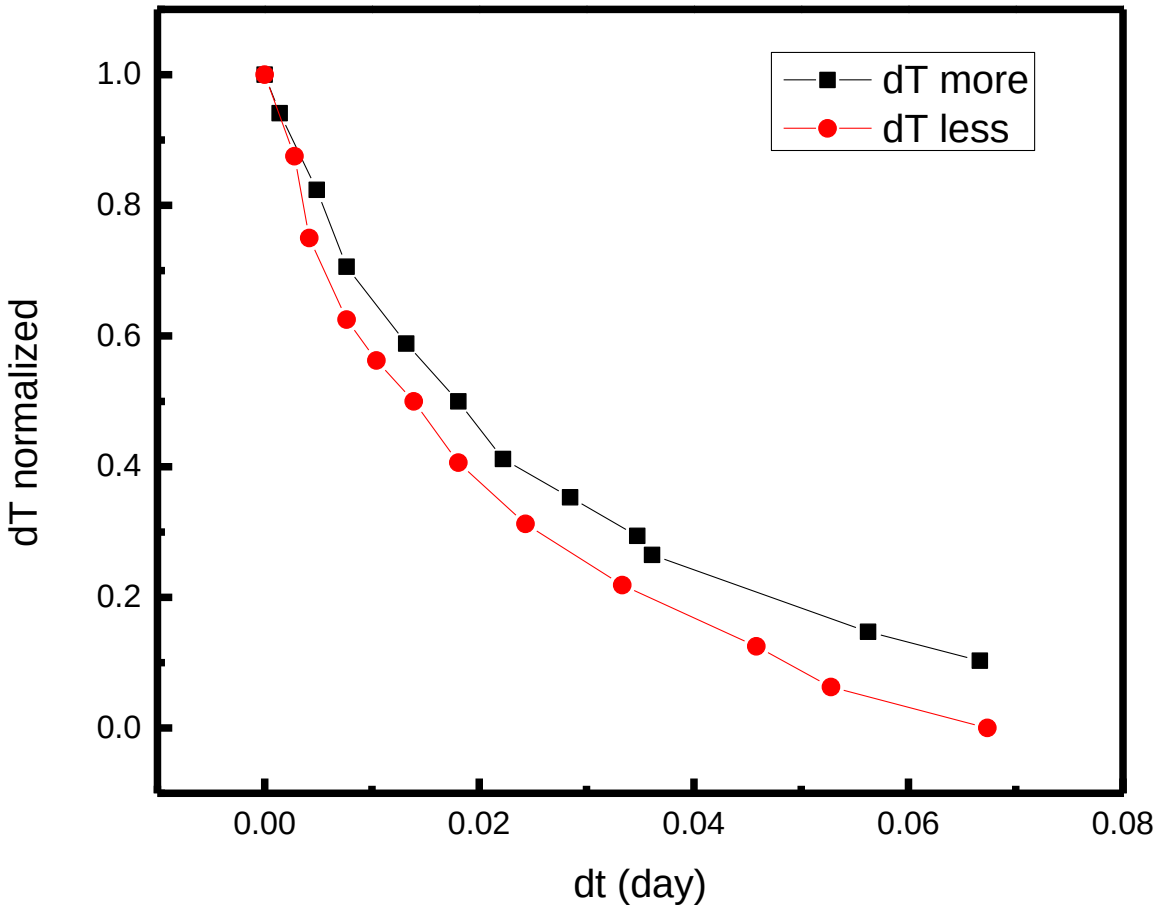
$$\tau = RC = \frac{d}{\sigma A} c_M \rho A d = \frac{c_M \rho}{\sigma} \boldsymbol{d^2}$$

Again, this scales with width^2.

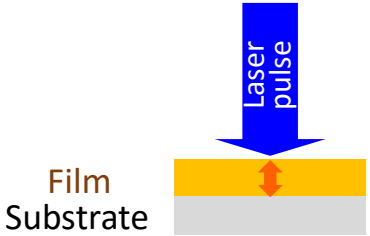
Some hands-on data

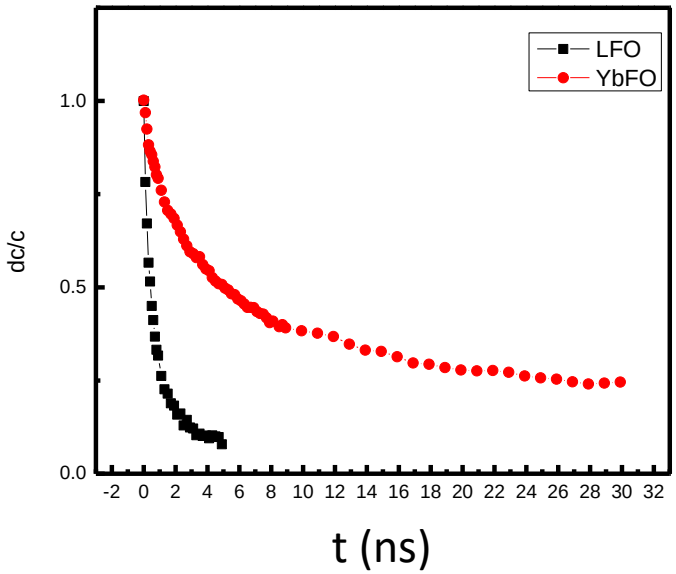
A simple experiment



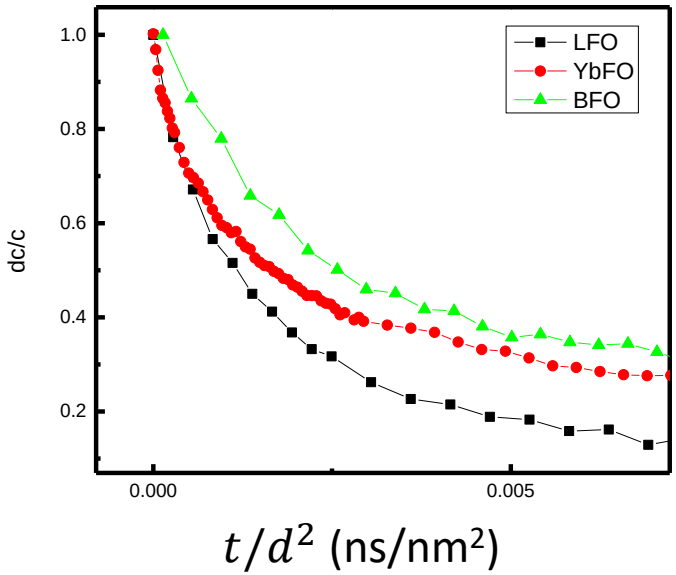


Time scale in thin films





20 nm
50 nm



20 nm
50 nm
35 nm

For the previous water cooling experiment:

$\boldsymbol{\tau/d^2 = 0.003}$ **(ns/nm²)**

$\tau = 1080\text{s}$

$d = 2$ cm

Conclusion

- We discussed the scaling rule of the thermal conduction, which is proportional to the thickness^2.
- A few examples (microscopic and macroscopic) are given, which can be unified using the scaling rule.